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PROBABILITY OF A PURE EQUILIBRIUM POINT IN n-PERSON GAMES

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PREFACE

This Memorandum continues Project RAND's program of research into the theory of games and its applications. It extends recent work done on n-person games and reported in RM-5543-PR, RM-5567-PR, and RM-5438-PR.

The result obtained herein, namely, that a large nonzero-sum n-person game chosen at "random" is likely to have a pure strategy equilibrium point, may have important implications. Every game has a mixed strategy equilibrium point, but it is not clear how or why one would ever use such a solution concept in a real-world situation. Not only is an optimal mixed strategy very difficult to compute, but decisionmakers are reluctant to make operational use of the notion because it means leaving the decision to chance. This Memorandum indicates that if the players have many strategies, a mixed strategy is rarely an optimal one. Thus many game theory models may take on additional significance.

SUMMARY

A "random" n-person noncooperative game—the game that prohibits communication and therefore coalitions among the n-players—is shown to have a pure strategy solution with a high probability. A solution of a game is an equilibrium point or a set of strategies, one for each player, such that if $n - 1$ players use their equilibrium strategies then the n-th player has no reason to deviate from his equilibrium strategy. It is shown that the probability of a solution in pure strategies for large random games converges to $1 - \frac{1}{e}$ for all $n \geq 2$.

CONTENTS

1. INTRODUCTION

The concept of a solution, or optimal strategy, frequently used for an n-person noncooperative game is the equilibrium strategy or equilibrium point. In order to assure the existence of a solution it is necessary to introduce mixed strategies. Except for the 2-person zero-sum game, however, it is generally very difficult to compute an optimal mixed startegy. Further, the decision-maker is reluctant to accept the operational notion of a mixed strategy.

These limitations of mixed strategies lead naturally to the hope that mixed strategy solutions are rarely required, or that a game chosen at random will in fact possess a pure strategy solution. For a 2-person zero-sum game this hope is not fulfilled; for large matrices in such games it is almost certain that the solution will be a mixed strategy, or the chance of a pure strategy solution is almost negligible.

It was conjectured that the optimal strategy of an n-person game would have a similar property. The present paper shows that with respect to solution, the n-person game is different from the 2-person zero-sum game. It is shown that the probability of a solution in pure strategies is quite large. in fact converging to

$$1 - e^{-1} = .632^{+}$$

for large games. Further, this result is the same regardless of the number of players, two or more.

2. GAMES AND TRUNCATIONS

In the normal form of an n-person noncooperative game the i-th player ($i \leq n$) has m_i strategies which we label $u_i (1 \leq u_i \leq m_i)$. A play of a game can be represented by an n-vector $U = (u_1, u_2, \ldots, u_n)$, giving us $\prod_{i=1}^{n} m_i =$ [illegible] possible plays. For each play U and each player i there exists a payoff $M_i(U)$, representing the payoff to the i-th player for the play U. There are therefore n[illegible] payoffs.

We now define a truncation of a play with respect to the i-th player to be an $n - 1$ vector

$$U_i = (u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n).$$

A truncation of a play leaves out the i-th player's strategy or

$$U = (U_i, u_i).$$

3. EQUILIBRIUM POINT

Nash [1] first introduced the notion of an equilibrium point, and he showed that every game possesses such a point in mixed strategies. An n-vector of pure strategies $U^* = (u_1^*, u_2^*, \ldots, u_n^*)$ is an equilibrium point in pure strategies if for each $i \leq n$ and $u_i \leq m_i$,

$$M_i(U^*) \geq M_i(U_i^*, u_i). \tag{1}$$

Equivalently, we have, for each $i \leq n$,

$$M_i(U^*) = \operatorname*{Max}_{u_i \leq m_i} M_i(U_i^*, u_i). \tag{2}$$

If the above condition is satisfied, U^* will be referred to as a pure equilibrium point or PE solution or just PE. For a 2-person zero-sum game a PE solution is the same as a saddle-point. We also call a PE point a solution of the n-person game.

4. RANDOM GAMES

It is well-known that PE solutions are rare for 2-person zero-sum games. For example, the probability that a "random" 2-person zero-sum game has a PE solution is

$$\frac{m_1!\ m_2!}{(m_1+m_2-1)!}\ .$$

This result exhibits the need for mixed strategies, even if the number of strategies for each player isn't very large in the 2-person zero-sum game.

It is natural to inquire about the need for mixed strategies in arbitrary n-person games. Is it likely that we can get by with pure strategies? To answer this inquiry we analyze "random games."

We define a random n-person game by the following properties:

(i) The n payoffs $M_i(U)$, are independent random variables.

(ii) For each i, the payoffs $M_i(U)$ have the same continuous probability distribution.

From the above definition of a random game it follows that the n payoffs are distinct in such a game. Further, the probability that a random n-person game has a PE solution is now well-defined.

Let E(U) be the event that U is a PE solution of the game. Define the following probabilities

$$S_1 = \sum_j \Pr\{E(U^j)\}, \quad S_2 = \sum_{j,k} \Pr\{E(U^j)E(U^k)\},$$

$$S_3 = \sum_{j,k,\ell} \Pr\{E(U^j)E(U^k)E(U^\ell)\}, \ \ldots \quad .$$

Let $P_n(m_1, m_2, \ldots, m_n)$ be the probability that a random n-person game, where the n players have $m_1, m_2, \ldots, m_n$ strategies, respectively, has at least one PE solution. Then

$$P_n(m_1, m_2, \ldots, m_n) = \Pr\{\sum_U E(U)\}$$

Then by the so-called method of inclusion and exclusion

$$P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1} (-1)^{t+1} S_t .$$

Since the events are equally-likely, we have that

$$S_t = \frac{N_t}{t} ,$$

where N_t is the cardinality of the family of all sets which have t equilibrium points, or

(3) $$P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1} (-1)^{t+1} N_t \cdot{}^{-t} .$$

5. EXISTENCE OF t EQUILIBRIUM POINTS

In order to determine N_t we shall derive a condition that a game have t equilibrium points. Our definition of equilibrium point and random game yields the following

Theorem 1. A necessary and sufficient condition that $U^1, U^2, \ldots, U^t$ are t equilibrium points of an n-person game is that

$$U_i^1, U_i^2, \ldots, U_i^t \text{ are distinct for each } i \leq n.$$

Proof: Suppose

$$U_i^1 = U_i^2 .$$

Then since U^1 and U^2 are equilibrium points

$$M_i(U^1) = \max_{u_i \leq m_i} M_i(U_i^1, u_i)$$

$$= \max_{u_i \leq m_i} M_i(U_i^2, u_i) = M_i(U^2),$$

contradicting the implication that all n payoffs are distinct.

Since the U's are n-vectors and the U_i's are (n−1)-vectors, the theorem states that each pair of U's must differ in at least two of their n-components in order to be PE solutions.

6. EQUILIBRIUM POINTS IN TWO-PERSON GAMES

If $n = 2$, a play of the game can be represented by a 2-vector $U = (\alpha, \beta)$. In order for $(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^t, \beta^t)$ to be t equilibrium points, then from Theorem 1 it follows that

$$\alpha^1, \alpha^2, \ldots, \alpha^t \text{ are distinct}$$

and

$$\beta^1, \beta^2, \ldots, \beta^t \text{ are distinct.}$$

To compute N_t, we observe that t distinct α's can be chosen in $\binom{m_1}{t}$ ways and t distinct β's can be chosen in $\binom{m_2}{t}$ ways, and then the two sets can be paired off in $t!$ ways. Thus

$$N_t = \binom{m_1}{t}\binom{m_2}{t}t! ,$$

and

$$P_2(m_1, m_2) = \sum_{t=1} (-1)^{t+1}\binom{m_1}{t}\binom{m_2}{t}t!(m_1m_2)^{-t} . \tag{4}$$

This result was first obtained by K. Goldberg, A. J. Goldman and M. Newman 2 . They also obtained the asymptotic value of $P_2(m_1, m_2)$.

7. EQUILIBRIUM POINTS IN THREE PERSON-GAMES

If $n = 3$, it is convenient to decompose the set of [illegible] $= m_1m_2m_3$ points into m_1m_2 sets of the form S_{ij}. Each member $U = (u_1, u_2, u_3)$ of S_{ij} is such that $u_1 = i$, $u_2 = j$, $u_3 \leq m_3$. Thus each set S_{ij} contains m_3 points. Now each S_{ij} can contain at most one equilibrium point. Therefore N_t can be determined by the following process:

(i) Choose t sets $S_{i_1j_1}, S_{i_2j_2}, \ldots, S_{i_tj_t}$ from the m_1m_2 sets S_{ij}.

(ii) Choose one member from each of these t sets so that the t choices are PE points.

Let $\mu(t)$ be the number of ways of choosing t PE points satisfying (ii) above—i.e., $\mu(t)$ is the number of ways of choosing t equilibrium points from t given sets $S_{i_1j_1}, S_{i_2,j_2}, \ldots, S_{i_tj_t}$. We have then

$$N_t = \binom{m_1m_2}{t}\mu(t). \tag{5}$$

For example, if $t = 1$, $\mu(1) = m_3$, and

$$N_1 = (m_1m_2)m_3 = \text{[illegible]} .$$

If $t = 2$, we have

$$\mu(2) = m_3^2 \qquad \text{if} \quad i_1 \neq i_2,\ j_1 \neq j_2$$

$$\mu(2) = (m_3 - 1)m_3 \quad \text{if} \quad i_1 = i_2 \ \text{or} \ j_1 = j_2$$

Let μ_t represent the number of ways of choosing t equilibrium points from the t sets $S_{i_1j_1}$, $S_{i_2j_2}$, $\ldots$, $S_{i_tj_t}$ given that $t-1$ equilibrium points have been chosen from the $t-1$ sets $S_{i_1j_1}$, $S_{i_2j_2}$, $\ldots$, $S_{i_{t-1}j_{t-1}}$. Then we have $\mu_1 = m_3$ and

(6) $$\mu(t) = \mu_t\mu(t-1) = \prod_{k=1}^{t} \mu_k .$$

It is evident that μ_t also represents the number of ways of choosing the t-th PE from the set $S_{i_tj_t}$ given that (t–1) PE points have been chosen from the (t–1) sets $S_{i_1j_1}$, $S_{i_2j_2}$, $\ldots$, $S_{i_{t-1}j_{t-1}}$. Hence μ_t must be bounded as follows:

(7) $$m_3 - t + 1 \leq \mu_t \leq m_3 .$$

For example, to compute μ_2, we have

$$\mu_2 = m_3 \quad \text{if } i_1 \neq i_2, \; j_1 \neq j_2$$

$$= m_3 - 1 \text{ if } i_1 = i_2 \text{ or } j_1 = j_2 .$$

Now the weights attached to the first value, m_3, is $(m_1 - 1)(m_2 - 1)$ while the weights attached to second value is $m_1 + m_2 - 2$. Therefore we have

$$\mu_2 = m_3 - \frac{m_1 + m_2 - 2}{m_1 m_2 - 1} \quad .$$

We can now compute

$$\mu(2) = m_3 \left(\frac{m_1 m_2 m_3 - m_1 - m_2 - m_3 + 2}{m_1 m_2 - 1}\right)$$

$$= m_3 \left(\frac{\pi - S + 2}{m_1 m_2 - 1}\right)$$

where $S = m_1 + m_2 + m_3$.

Substituting in (5) we get

$$N_2 = \binom{m_1 m_2}{2} \mu(2) = \frac{\pi(\pi - S + 2)}{2} \quad .$$

To compute μ_3 we need to examine four cases

$\mu_3 = m_3 - 2$ if $i_1 = i_2 = i_3$ and j_1, j_2, j_3 distinct,
or if $j_1 = j_2 = j_3$ and i_1, i_2, i_3 distinct

$\mu_3 = m_3 - 1$ if $i_1 = i_2 \neq i_3$, j_1, j_2, j_3 distinct
or if $j_1 = j_2 \neq j_3$, i_1, i_2, i_3 distinct

$\mu_3 = m_3$ if i_1, i_2, i_3 distinct, j_1, j_2, j_3 distinct

$\mu_3 = \frac{(m_3-1)^2}{m_3}$ if $i_1 = i_2 \neq i_3$ and $j_1 = j_3 \neq j_2$
or if $j_1 = j_2 \neq j_3$ and $i_1 = i_3 \neq i_2$.

The weights associated with each of the four above values of μ_3 are, respectively

$$(m_1 - 1)(m_1 - 2) + (m_2 - 1)(m_2 - 2)$$

$$2(m_1 - 1)(m_2 - 1)^2 + 2(m_1 - 1)^2(m_2 - 1)$$

$$(m_1 - 1)(m_2 - 1)(m_1m_2 - m_1 - m_2)$$

$$2(m_1 - 1)(m_2 - 1)$$

The sum of the above weights is $(m_1m_2 - 1)(m_1m_2 - 2)$.

Using the above weights and values of μ_3 we obtain an average value of μ_3 as a function of m_1, m_2, m_3. In particular, if $m_1 = m_2 = m_3 = m$, this average value is

$$\mu_3 = m - \frac{2(2m^3 - 5m + 1)}{m(m + 1)(m^2 - 2)} \quad .$$

In terms of the above value of μ_3, we can now evaluate

$$N_3 = \binom{m_1m_2}{3}\mu(3)$$

$$= \frac{m_1m_2 - 2}{3} N_2\mu_3$$

$$= \frac{[illegible](\,[illegible] - 2m_3)(\,[illegible] - S + 2)}{6m_3} \mu_3 \; .$$

In a similar manner we can compute recursively the values of N_t and then compute the required probability

$$
\begin{aligned}
P_3\,(m_1, m_2, m_3) &= \sum_{t=1} (-1)^{t+1} N_t n^{-t} \\
&= \sum_{t=1} (-1)^{t+1} \binom{m_1 m_2}{t} \prod_{k=1}^{t} \frac{\mu_k}{n} \,.
\end{aligned}
\tag{8}
$$

It is of interest to determine the asymptotic value of $P_3(m_1, m_2, m_3)$ as the number of strategies increase for each player. We note that the absolute value of the t-th term of the series for P_3 is

$$
\begin{aligned}
N_t n^{-t} &= \binom{m_1 m_2}{t} \prod_{k=1}^{t} \frac{\mu_k}{m_1 m_2 m_3} \\
&= \frac{1}{t!} \prod_{k=1}^{t} \frac{(m_1 m_2 - k+1)\mu_k}{m_1 m_2 m_3} \,,
\end{aligned}
$$

where

$$
m_3 - k+1 \leq \mu_k \leq m_3, \qquad k \leq m_3 \,.
$$

Hence we have for $k \leq t \leq m_1 m_2$

$$
\left(1 - \frac{k-1}{m_1 m_2}\right)\left(1 - \frac{k-1}{m_3}\right) \leq \frac{(m_1 m_2 - k+1)\mu_k}{m_1 m_2 m_3} \leq \left(1 - \frac{k-1}{m_1 m_2}\right) .
$$

Thus

$$\prod_{k=1}^{t}(1-\frac{k-1}{m_1m_2})(1-\frac{k-1}{m_3}) \leq \prod_{k=1}^{t}\frac{(m_1m_2-k+1)\mu_k}{m_1m_2m_3} \leq \prod_{k=1}^{t}(1-\frac{k-1}{m_1m_2}).$$

From the above inequality, it follows that

$$\lim_{m_1,m_2,m_3\to\infty}\prod_{k=1}^{t}(\frac{m_1m_2-k+1)\mu_k}{m_1m_2m_3} = 1$$

or

$$\lim_{m_1,m_2,m_3\to\infty} N_t\cdot^{-t} = \frac{1}{t!} .$$

Hence we get the asymptotic value of the probability

$$\lim_{m_1,m_2,m_3\to\infty} P_3(m_1,m_2,m_3) = \sum_{t=1}\frac{(-1)^{t+1}}{t!} = 1-e^{-1}.$$

8 PURE EQUILIBRIUM POINTS IN n-PERSON GAMES

We now evaluate the probability of a PE solution in a random n-person game, where the i-th player has m_i strategies. In such a game the set of $= m_1 m_2 \cdots m_n$ points can be decomposed into $m_1 m_2 \cdots m_{n-1} = M$ sets of the form $S_{i_1 i_2 \cdots i_{n-1}}$ where each set contains m_n points. Each member $U = (u_1, u_2, \ldots, u_n)$ of $S_{i_1 i_2 \cdots i_{n-1}}$ is such that $u_1 = i_1$, $u_2 = i_2$, $\ldots$, $u_{n-1} = i_{n-1}$, and $u_n = m_n = m$. Thus each set contains m points.

From Theorem 1 it follows that each set $S_{i_1 i_2 \cdots i_{n-1}}$ can contain at most one PE point. Therefore choosing t equilibrium points from the points is equivalent to choosing t of the M sets and then choosing one point from each of these t chosen sets. Again, let $\mu(t)$ be the number of ways of choosing t equilibrium points from the t chosen sets (we emphasize that only one point may be chosen from each set). Then, we have

$$N_t = \binom{M}{t}\mu(t) \quad .$$

As in the previous section let μ_t be the number of ways of choosing t equilibrium points from the t sets

$$S_{i_1 i_2 \cdots i_{n-1}} = T_1, \; S_{j_1 j_2 \cdots j_{n-1}} = T_2, \; \ldots, \; S_{\ell_1 \ell_2 \cdots \ell_{n-1}} = T_t .$$

given that $t - 1$ equilibrium points have been chosen from the $t - 1$ sets $T_1, T_2, \ldots, T_{t-1}$. It follows that

$$(9) \qquad \mu(t) = \mu_t \, \mu(t-1) = \prod_{k=1}^{t} \mu_k \, .$$

In making the above choices we need to choose the t-th PE point from the set T_t which contains m points. We thus have the following inequality

$$(10) \qquad m - t + 1 \le \mu_t \le m \, .$$

The required probability of a PE point in the random game is given by

$$P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1} (-1)^{t+1} N_t \cdot^{-t}$$

$$= \sum_{t=1} (-1)^{t+1} \binom{M}{t} (Mm)^{-t} \prod_{k=1}^{t} \mu_k \, .$$

We may write this probability as

$$(11) \qquad P_n(M, m) = \sum_{t=1} (-1)^{t+1} \binom{M}{t} M^{-t} \prod_{k=1}^{t} \left(\frac{\mu_k}{m}\right) \, .$$

where μ_k is a function of $m_1, m_2, \ldots, m_{m-1}, m$.

For each M and m we can compute $P_n(M,m)$ by first computing $\frac{\mu_k}{m}$ where $k \le t$. From the definition of μ_t

we have $\frac{\mu_1}{m} = 1$. In order to get $\frac{\mu_2}{m}$ we note that

$$\mu_2 = m \quad \text{if} \quad i_1 \neq j_1, \; i_2 \neq j_2, \; \ldots, \; i_{n-1} \neq j_{n-1}$$

$$\mu_2 = m-1 \quad \text{if} \quad i_1 = j_1, \text{ or } i_2 = j_2, \; \ldots, \text{ or } i_{n-1} = j_{n-1} \; .$$

The weight associated with $\mu_2 = m$ is $(m_1 - 1)\,(m_2 - 1) \cdots (m_{n-1} - 1) = D$. The weight associated with $\mu_2 = m - 1$, is $\pi - D - 1$. We thus get

$$\mu_2 = m - \frac{\pi - D - 1}{M - 1} \; ,$$

and

$$\frac{\mu_2}{m} = 1 - \frac{\pi - D - 1}{m(M-1)} = 1 - \frac{mM - D - 1}{mM - m} \; .$$

In a similar manner we can compute $\frac{\mu_3}{m}, \frac{\mu_4}{m}, \ldots, \frac{\mu_t}{m}$ and then obtain P_n. Of course, the computation of $\frac{\mu_k}{m}$ becomes more cumbersome with each value of k. However, P_n has an asymptotic value given by

<u>Theorem 2</u>. <u>For all n-person games</u> $(n \geq 2)$

$$\lim_{\substack{M \to \infty \\ m \to \infty}} P_n(m_1, m_2, \ldots, m) = 1 - e^{-1}.$$

<u>Proof</u>: Equation (11) can be written as

$$P_n(M,m) = \sum_{t=1} \frac{(-1)^{t+1}}{t!} \prod_{i=1}^{t-1} (1 - \frac{i}{M}) \prod_{k=1}^{t} \frac{\mu_k}{m} .$$

Hence we have

$$(12) \qquad P_n(M,m) - (1 - e^{-1}) = \sum_{t=0} \frac{(-1)^t}{t!} \left[1 - \prod_{i=1}^{t-1} (1 - \frac{i}{M}) \prod_{k=1}^{t} \frac{\mu_k}{m}\right] .$$

Now let

$$\lambda_t(M,m) = \prod_{i=1}^{t-1} (1 - \frac{i}{M}) \prod_{k=1}^{t} \frac{\mu_k}{m} .$$

It is clear from (10) that for all t,

$$0 \leq \lambda_t(M,m) \leq 1 .$$

Now for all $i \leq T \leq M$ we have

$$1 \geq 1 - \frac{i}{M} \geq 1 - \frac{T}{M} .$$

Hence

$$\prod_{i=1}^{t-1} (1 - \frac{i}{M}) > (1 - \frac{T}{M})^{t-1} > (1 - \frac{T}{M})^T \quad \text{for} \quad t \leq T \leq M .$$

We also have from (10) that

$$\frac{\mu_k}{m} \geq 1 - \frac{k-1}{m} > 1 - \frac{t}{m} > 1 - \frac{T}{m} \quad \text{for} \quad k < t < T.$$

Therefore

$$\prod_{k=1}^{t} \frac{\mu_k}{m} > (1 - \frac{T}{m})^T \qquad\qquad t < T < m.$$

Substituting the above inequalities in the definition of $\lambda_t(M,m)$ we have

$$\begin{aligned} \lambda_t(M,m) &\geq (1 - \frac{T}{M})^T \ (1 - \frac{T}{m})^T \quad \text{for} \quad t < T < M \\ & \qquad\qquad\qquad\qquad\qquad\quad \text{and} \quad t < T < m \\ &\geq (1 - \frac{T^2}{M}) \ (1 - \frac{T^2}{m}) \\ &\geq (1 - \frac{T^2}{M} - \frac{T^2}{m}) \quad . \end{aligned}$$

Now T is arbitrary but $T < M$ and $T < m$. Suppose we restrict T so that $T^3 < M$, and $T^3 < m$, then $\frac{T^2}{M} < \frac{1}{T}$ and $\frac{T^2}{m} < \frac{1}{T}$, and we obtain the inequality

$$\begin{aligned} 1 \geq \lambda_t(M,m) \geq 1 - \frac{2}{T} \quad &\text{for} \quad t < T < T^3 < M \\ &\text{and} \quad t < T < T^3 < m \ . \end{aligned}$$

Or

$$0 \leq 1 - \lambda_t(M,m) < \frac{2}{T} \quad \text{for} \quad t < T < T^3 < M$$
$$\text{and} \quad t < T < T^3 < m \,.$$

Returning to (12) we have

$$\left| P_n(M,m) - (1 - e^{-1}) \right| \leq \sum_{t=0}^{T} \frac{1}{t!} \left(\frac{2}{T}\right) + \left| \sum_{t>T} \frac{(-1)^t}{t!} [1 - \lambda_t(M,m)] \right|$$

$$\leq \frac{2}{T}e + \left| \sum_{t>T} \frac{(-1)^t}{t!} [1 - \lambda_t(M,m)] \right| \,.$$

The second term represents the "tail" of a converging alternating series, hence there exists $\delta > 0$ such that

$$\sum_{t>T} \frac{(-1)^t}{t!} [1 - \lambda_t(M,m)] < \delta$$

giving us

$$\left| P_n(M,m) - (1 - e^{-1}) \right| < \frac{2}{T}e + \delta.$$

Let $T > \frac{2\delta}{\varepsilon}$, then

$$\left| P_n(M,m) - (1 - e^{-1}) \right| < 2\delta,$$

which proves the theorem. It is of interest to note that Theorem 2 requires only that two of the n players sets of strategies be infinite.

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10. ABSTRACT

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